Math 2020 Tut 10

Green's thm: $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \vec{\nabla}\times\vec{F}\cdot\vec{k}\,dA$

$$\oint_C \vec{F}\cdot\vec{n}\cdot ds = \iint_R \nabla\cdot\vec{F}\,dA$$

Q1: Lecture 18: i) $\vec{\nabla}\times(\vec{\nabla}f)=\vec{0}$

ii) $\vec{F}$ conservative $\Rightarrow$ curl $\vec{F} = \nabla\times\vec{F}=0$

iii) $\vec{\nabla}\cdot\vec{\nabla}\times\vec{F}=0$

Ans: i) $\vec{\nabla}\times\nabla f = \vec{\nabla}\times(f_x\vec{i}+f_y\vec{j}+f_z\vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix}$$

$$= i(f_{zy}-f_{yz}) - \vec{j}(f_{zx}-f_{xz}) + \vec{k}(f_{yx}-f_{xy})$$

$$= 0$$

ii) Theorem 10

iii) $\nabla\cdot\nabla\times\vec{F} = \begin{vmatrix} \partial_x & \partial_y & \partial_z \\ \partial_y & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$

Q2: Let $\gamma(t) = (u(t), v(t))$ $\quad 0 \le t \le 1$ be a loop,

with $\gamma(t) \ne 0$. Let $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$

Let $A(t) = \int_0^t \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt$

$B(t) = \frac{u(t) + i v(t)}{|u(t) + i v(t)|}$

a) Show $B'(t) = i B(t) A'(t)$

b) Show that $B(t) \exp(-iA(t)) \equiv \text{constant}$

c) Show that $\frac{1}{2\pi} \int_\gamma \vec{F} \cdot \vec{T} \cdot ds \in \mathbb{N}$.

Ans: a) $A'(t) = \vec{F}(\gamma(t)) \cdot \gamma'(t) = \frac{-vu' + uv'}{u^2 + v^2}$

$B = \frac{u + iv}{\sqrt{u^2+v^2}}$, $\quad B' = \frac{(u+iv)'\sqrt{u^2+v^2} - (u+iv)\left(\sqrt{u^2+v^2}\right)'}{u^2+v^2}$

$$= \frac{1}{u^2+v^2}\left[(u'+iv')\sqrt{u^2+v^2} - (u+iv)\frac{uu'+vv'}{\sqrt{u^2+v^2}}\right]$$

$$= \frac{1}{(u^2+v^2)^{3/2}}\left[(u'+iv')(u^2+v^2) - (u+iv)(uu'+vv')\right]$$

while $(u'+iv')(u^2+v^2) - (u+iv)(uu'+vv')$

$= (u^2u' + v^2u' - u^2u' - uvv') + i(u^2v' + v^2v' - vuu' - v^2v')$

$= v(vu' - uv') + iu(uv' - vu')$

$= (-v + iu)(uv' - vu')$

$= i(u + iv)(uv' - vu')$

So $\quad B' = \frac{1}{(u^2+v^2)^{3/2}}\left[i(u+iv)(uv'-vu')\right]$

$$= i\, \frac{u+iv}{|u+iv|}\, \frac{uv'-vu'}{u^2+v^2} = iBA'$$

b) $\frac{d}{dt}\left(B\exp(-iA)\right) = B'\exp(-iA) + B(\exp(-iA))' = B(iA')\exp(-iA) + B\exp(-iA)(-iA') = 0$

c) From b) $B(0)\exp(-iA(0)) = B(1)\exp(-iA(1))$

$B(0) = B(1) \neq 0 \Rightarrow \exp(-iA(0)) = \exp(-iA(1))$

$$A(0) = \int_0^0 \vec{F}\cdot\gamma'\,dt = 0$$

$\Rightarrow \quad 1 = \exp(-iA(0)) = \exp(-iA(1))$

$\Rightarrow \quad A(1) = 2n\pi \quad , \quad n \in \mathbb{N}$

i.e. $\quad \int_\gamma \vec{F}\cdot\vec{T}\,ds = \int_0^1 \vec{F}\cdot\gamma'\,dt = 2n\pi$

Surface area of sphere of radius 1

Ans: 1. Find a parametrization

$$(x,y,z) = (\cos\phi\cos\theta,\ \cos\phi\sin\theta,\ \sin\phi)$$

2. Find r_ϕ, r_θ

$$r_\phi = (-\sin\phi\cos\theta,\ -\sin\phi\sin\theta,\ \cos\phi) \qquad 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi$$

$$r_\theta = (-\cos\phi\sin\theta,\ \cos\phi\cos\theta,\ 0)$$

3. Find $|r_\phi \times r_\theta|$

$$r_\phi \times r_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\phi\cos\theta & -\sin\phi\sin\theta & \cos\phi \\ -\cos\phi\sin\theta & \cos\phi\cos\theta & 0 \end{vmatrix}$$

$$= -\cos^2\phi\cos\theta\,\vec{i} - \cos^2\phi\sin\theta\,\vec{j} - \sin\phi\cos\phi\,\vec{k}$$

$$|r_\phi \times r_\theta|^2 = \cos^4\phi\cos^2\theta + \cos^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi$$

$$= \cos^4\phi + \sin^2\phi\cos^2\phi$$

$$= \cos^2\phi$$

$$|r_\phi \times r_\theta| = |\cos\phi|$$

Find $A = \iint_R |r_\phi \times r_\theta|\, d\phi\, d\theta$

$$A = \int_0^{2\pi}\int_0^{\pi} |\cos\phi|\, d\phi\, d\theta = 2\int_0^{2\pi}\int_0^{\pi/2} \cos\phi\, d\phi\, d\theta$$

$$= 2\int_0^{2\pi} [\sin\phi]_0^{\frac{\pi}{2}}\, d\theta$$

$$= 2\int_0^{2\pi} d\theta = 4\pi$$

Q4: Let $\vec{r} = \vec{F}(u,v)$, $(u,v) \in R$

be a parametrization of a surface S.

Let $E(u,v) = \langle \vec{r}_u , \vec{r}_u \rangle$, $F(u,v) = \langle \vec{r}_u , \vec{r}_v \rangle$, $G(u,v) = \langle \vec{r}_v , \vec{r}_v \rangle$

Show that the area of S is given by

$$\text{Area}(S) = \int_R \sqrt{EG - F^2}\, du\, dv$$

Ans: $\text{Area}(S) = \int_R \| r_u \times r_v \| \, du\, dv$

While $\| r_u \times r_v \| = \|r_u\| \|r_v\| \sin\theta$

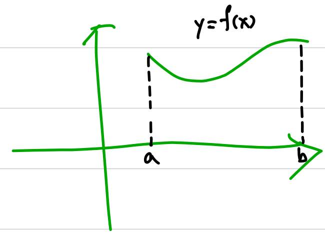
$$\| r_u \times r_v \|^2 = \|r_u\|^2 \|r_v\|^2 (1 - \cos^2\theta)$$

$$= \|r_u\|^2 \|r_v\|^2 - \|r_u\|^2 \|r_v\|^2 \cos^2\theta$$

$$= \|r_u\|^2 \|r_v\|^2 - \langle r_u , r_v \rangle^2$$

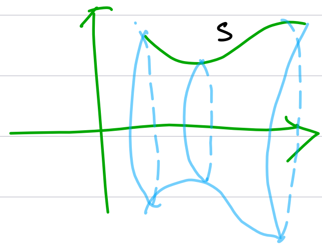
$$= EG - F^2$$

Q5 (Surface of revolution)



The graph of the function $y=f(x)$ $(f>0)$ in the x-y plane is rotated about the x-axis to form a surface S in $\mathbb{R}^3$. Show

$$\text{Area}(S) = 2\pi\int_a^b f(u)\sqrt{1+f'(u)^2}\,du$$



(In general $2\pi\int_a^b |f(u)|\sqrt{1+f'(u)^2}\,du$

Ans: 1. Find a parametrization

$$\vec{r}(u,\theta) = (u, f(u)\cos\theta, f(u)\sin\theta)$$

$$a\le u\le b, \quad 0\le\theta\le 2\pi$$

2. Find $|\vec{r}_u\times\vec{r}_v|$

$$\vec{r}_u = (1, f'(u)\cos\theta, f'(u)\sin\theta)$$

$$\vec{r}_v = (0, -f(u)\sin\theta, f(u)\cos\theta)$$

We try use Q4: $E = 1+f'(u)^2$, $F=0$, $G=f(u)^2$

So $|\vec{r}_u\times\vec{r}_v| = \sqrt{EG-F^2} = \sqrt{f(u)^2(1+f'(u)^2)} = f(u)\sqrt{1+f'(u)^2}$

$$\Rightarrow \text{Area}(S) = \int_a^b\int_0^{2\pi} f(u)\sqrt{1+f'(u)^2}\,d\theta\,du$$

$$= 2\pi\int_a^b f(u)\sqrt{1+f'(u)^2}\,du$$

Remark: If we let $\alpha(u) = (u, f(u))$ be a parametrization of the graph of f.

Then $\text{Area}(S) = 2\pi\int_\alpha f\,ds$

$$= \int_\alpha \text{circumference of } C_u\,ds$$